times, and with every conceivable modification and check. Some few of them have already been published in the 'Chemical News,' but are here referred to again for the sake of comprehensiveness.

At present the writer does not venture to put forth any definite theory respecting the presence and nature of the nuclei which are so universally diffused throughout the atmosphere; but when it is considered how much sodic chloride is constantly present in the air, and what quantities of sulphurous acid are evolved daily, which becomes partly converted into sulphuric acid, the presence of particles of sodic sulphate in the air would not be surprising; and that it does exist is proved by drawing air through water and finding comparatively large quantities in the solid matter arrested by water.

Sodic sulphate solutions, too, crystallize on exposure much more readily than those of any other salt. The other salts which form supersaturated solutions are certainly less diffused than sodic sulphate.

XXI. "Note relating to the Attraction of Spheroids." By I. Todhunter, M.A., F.R.S. Received May 16, 1872.

In a memoir on the Attraction of Spheroids, published in the 'Connaissance des Tems' for 1829, Poisson showed that certain important formulæ were true up to the *third* order inclusive of the standard small quantity. The object of this note is to establish the truth of the formulæ for *all* orders of the small quantity.

1. Suppose we require the value of the potential of a homogeneous body at any assigned point. Take a fixed origin inside the body; let r', θ', ψ' denote the polar coordinates of any point of the body; and let r, θ, ψ be the polar coordinates of the assigned point; and, as usual, put μ' for $\cos\theta'$, and μ for $\cos\theta$. The density may be denoted by unity.

Then the potential V is given by the equation

$$V=\int\int\int\frac{r'^2\,dr'\,d\mu'\,d\psi'}{\sqrt{(r^2+r'^2-2rr'\lambda)}},$$

where

$$\lambda=\mu\mu'+\sqrt{(1-\mu^2)}\sqrt{(1-\mu'^2)}\cos(\psi'-\psi).$$

The integration must extend over the whole body.

2. Suppose that r is greater than the greatest value of r'; then $(r^2+r'^2-2rr'\lambda)^{-\frac{1}{2}}$ can be expanded in a convergent series of powers of $\frac{r'}{r}$. Thus, for example, let the body be an ellipsoid, and take the centre as the origin; let a, b, c denote the semiaxes in descending order of magnitude. Then, if r is greater than a, the expansion may be effected in the manner just stated; and so a convenient expression may be obtained for the potential of an ellipsoid on an external particle. This expression, however, is not demonstrated to hold for every external particle, but only for

those which make r greater than a. It is obvious that there may be external particles for which r is less than a; and for these the process cannot be considered satisfactory, since it involves the use of a divergent series.

3. Still it has been usual with writers on the Attraction of Spheroids and the Figure of the Earth to leave this point unexamined. They, in fact, assume that formulæ which are demonstrated on a certain condition are true, even when that condition does not hold; so that, for example, an expression obtained strictly for the potential of an ellipsoid on an external particle when r is greater than a, is assumed to be true for any external particle.

4. Poisson, however, has drawn attention to the difficulty; his discussion of it is the main part of his elaborate memoir "Sur l'Attraction des Sphéroides," which was published in the 'Connaissance des Tems' for 1829. He shows that the ordinary formulæ, although obtained in an inadequate manner, are really true *as far as the terms of the order* α^3 *inclusive,* where α is the well-known standard small quantity of such investigations. I propose to extend his process so as to show that the result is true for all powers of α.

It will be necessary to give some preliminary transformations; this I shall do with brevity, referring to Poisson's memoir for detail.

5. It is convenient to separate V into two parts, one being the potential of a sphere of radius r, and the other the potential of the excess of the spheroid above the sphere; the word *excess* is here used in an algebraical sense, for the surface of the spheroid is not necessarily all external to that of the sphere. Thus we obtain

$$\mathrm{V}=\frac{4\pi r^3}{3}+\int\int\int_r^u \frac{r'^2\,d\mu'\,d\psi'\,dr'}{\sqrt{(r^2+r'^2-2\lambda r r')}}, \quad . \quad . \quad . \quad (1)$$

where u denotes the radius vector of the surface of the spheroid corresponding to the angles θ' and ψ'; so that the integration with respect to r' is to be taken between the limits r and u. The integration for μ' and ψ' may be considered to be taken over the surface of a sphere of radius unity; and we may denote an element of this surface by $d\omega'$, and use the symbol $\int d\omega'$ instead of $\int\int d\mu'\,d\psi'$.

6. Now, for those elements in the integral in (1) which have r' less than r, the radical must be expanded in powers of $\frac{r'}{r}$; and for those elements which have r' greater than r, the radical must be expanded in powers of $\frac{r}{r'}$. Thus we obtain

$$\mathrm{V}=\frac{4\pi r^3}{3}+\Sigma\frac{1}{r^{n+1}}\int\left(\int_r^u r'^{n+2}dr'\right)\mathrm{P}'_n d\omega'+\Sigma\, r^n\int\left(\int_r^u \frac{dr'}{r'^{n-1}}\right)\mathrm{P}'_n d\omega', \quad (2)$$

where P'_n denotes Laplace's coefficient of the nth order. In the second

term on the right-hand side of (2) the integration with respect to ω' is to extend over so much of the surface of a sphere of radius unity as corresponds to *negative* values of $u-r$; and in the third term the integration with respect to ω' is to extend over so much of the surface of the sphere as corresponds to *positive* values of $u-r$. By Σ is denoted a summation with respect to the integer n for all values from zero to infinity.

7. By adding a certain quantity to the second term on the right-hand side of (2), and subtracting the same quantity from the third term, we obtain, finally,

$$V=\frac{4\pi r^2}{3}+\Sigma\frac{1}{r^{n+1}}\int_0^{4\pi}\left(\int_r^u r'^{n+2}dr'\right)P'_n\,d\omega'-\Sigma\int U\,P'_n\,d\omega', \quad . \quad . \quad (3)$$

where U stands for

$$\frac{1}{r^{n+1}}\int_r^u r'^{n+2}dr'-r^n\int_r^u\frac{dr'}{r'^{n-1}}.$$

In the second term on the right-hand side of (3) the integration for ω' extends over the *whole surface* of the sphere of radius unity; and this I denote by explicitly putting the limits 0 and 4π. But in the third term the integration for ω' extends only over that portion of the surface which corresponds to positive values of $u-r$; and this I denote by leaving the limits unspecified.

8. For the rest of this paper the notation just explained will be strictly preserved. If the integration with respect to ω' extends over the whole surface of the sphere, the limits 0 and 4π will be expressed; if the integration extends only over that portion of the surface which corresponds to positive values of $u-r$, the limits will not be expressed.

9. The value of V obtained in (3) is quite general, but it is specially convenient for the case of an external particle. Poisson gives also another form which is specially convenient for the case of an internal particle.

It will be sufficient for us to confine ourselves to the case of an external particle, as the same process is readily applicable to the case of an internal particle.

10. For an external particle which is sufficiently remote, the third term on the right-hand side of (3) vanishes, because in this case $u-r$ is never positive; so that we have then simply

$$V=\frac{4\pi r^2}{3}+\Sigma\frac{1}{r^{n+1}}\int_0^{4\pi}\left(\int_r^u r'^{n+2}\,dr'\right)P'_n\,d\omega'.$$

Now what we have to show is that this formula will also hold for every external particle. In other words, it must be shown that for *any* external particle

$$\Sigma\int U\,P'_n\,d\omega'=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

11. Put z' for $u-r$. Then

$$U=\frac{1}{(n+3)\,r^{n+1}}\{(r+z')^{n+3}-r^{n+3}\}+\frac{r^n}{n-2}\{(r+z')^{-n+2}-r^{-n+2}\}$$

$$=\frac{n+2+n-1}{2}z'^2+\frac{(n+2)(n+1)-(n-1)n}{\lfloor 3}z'^3$$

$$+\frac{(n+2)(n+1)n+(n-1)n(n+1)}{\lfloor 4}z'^4$$

$$+\frac{(n+2)(n+1)n(n-1)-(n-1)n(n+1)(n+2)}{\lfloor 5}z'^5$$

$$+\ldots\ldots$$

$$=\frac{2n+1}{2}z'^2+\frac{2n+1}{3}z'^3+\frac{(2n+1)(n^2+n)}{\lfloor 4}z'^4+\ldots\ldots$$

12. Let ζ' be a discontinuous function of μ' and ψ', such that ζ' is always equal to z' when z' is positive, and always zero when z' is negative. Then, for all values of m, we have

$$\int z'^m\,P'_n\,d\omega'=\int_0^{4\pi}\zeta'^m\,P'_n\,d\omega'.$$

This is a very important step in Poisson's process; and he explains it with adequate care. We may suppose that ζ' is expressed by means of a series of Laplace's functions.

13. As we may also suppose ζ'^2 expanded in a series of Laplace's functions, it will follow, from the well-known properties of such functions, that

$$\Sigma(2n+1)\int_0^{4\pi}\zeta'^2\,P'_n\,d\omega'=4\pi\zeta^2, \quad \ldots\ldots \quad (5)$$

where ζ is the value of ζ' when $\theta'=\theta$ and $\psi'=\psi$. But, by supposition, ζ is zero. Hence

$$\Sigma(2n+1)\int_0^{4\pi}\zeta'^2\,P'_n\,d\omega'=0,$$

and therefore

$$\Sigma(2n+1)\int z'^2\,P'_n\,d\omega'=0.$$

In precisely the same manner we have

$$\Sigma(2n+1)\int z'^3\,P'_n\,d\omega'=0.$$

14. Thus far Poisson carries his process. His words are, on his page 368:—"Pour simplifier la question, on néglige ici les puissances de ζ' supérieures à la troisième, ou autrement dit, on borne l'approximation aux quantités de l'ordre α^3 inclusivement."

I am not certain whether Poisson himself had carried his investigation beyond this point. In the later part of his memoir he certainly implies

that results which partly depend on the present investigations are true for all powers of $u-r$. It seems, therefore, curious that he did not here explain how the terms which involve powers of z' above the third vanish, so as to make (4) absolutely true. To this we now proceed.

15. In Art. 11 we see that the coefficient of z'^4 is

$$\frac{(2n+1)(n^2+n)}{\underline{|4}}.$$

Hence we have to show that

$$\Sigma(2n+1)(n^2+n)\int_0^{4\pi}\zeta'^4\,\mathrm{P}'_n\,d\omega'=0.$$

Now, by the nature of Laplace's coefficients, we have

$$(n^2+n)\,\mathrm{P}'_n=-\frac{d}{d\mu'}\left\{(1-\mu'^2)\frac{d\mathrm{P}'}{d\mu'}\right\}-\frac{1}{1-\mu'^2}\frac{d^2\,\mathrm{P}'_n}{d\psi'^2}. \quad . \quad . \quad (6)$$

Hence, by two integrations by parts, we find that

$$\Sigma(2n+1)\ (n^2+n)\int_0^{4\pi}\zeta'^4\,\mathrm{P}'_n\,d\omega'=-\Sigma(2n+1)\int_0^{4\pi}\mathrm{P}'_n\nabla\zeta'^4 d\omega',$$

where, for abbreviation, ∇ is used to denote the operation which, as performed on P'_n, is expressed on the right-hand side of (6).

Then, in the same way as (5) is obtained, we have

$$\Sigma(2n+1)\int_0^{4\pi}\mathrm{P}'_n\nabla\zeta'^4\,d\omega'=4\pi\nabla\zeta^4\,;$$

and $\nabla\zeta^4$ is zero, for every term involves ζ^2 as a factor.

Hence, finally,

$$\Sigma(2n+1)\ (n^2+n)\int_0^{4\pi}\zeta'^4\,\mathrm{P}'_n d\omega'=0.$$

16. In Art. 11 it will be found that the coefficient of z'^5 is zero.

The coefficient of z'^6 is

$$\frac{(n+2)\,(n+1)\,n\,(n-1)\,\{n-2+n+3\}}{\underline{|6}},$$

that is,

$$\frac{(n+2)\,(n-1)\,(n^2+n)\,(2n+1)}{\underline{|6}},$$

that is,

$$\frac{(n^2+n-2)\,(n^2+n)\,(2n+1)}{\underline{|6}},$$

that is,

$$\frac{(n^2+n)^2-2(n^2+n)}{\underline{|6}}(2n+1).$$

Hence we have to show that

$$\Sigma(n^2+n)^2(2n+1)\int_0^{4\pi}\zeta'^6\,P'_n\,d\omega'-2\Sigma(n^2+n)(2n+1)\int_0^{4\pi}\zeta'^6\,P'_n d\omega'=0.$$

The second term we see vanishes by the process of Art. 15. As to the first term, we must apply that process twice; and we shall then transform this term into

$$\Sigma(2n+1)\int_0^{4\pi}P'_n\nabla(\nabla\zeta'^6)\,d\omega';$$

and, as before, this is equal to $4\pi\nabla(\nabla\zeta^6)$, which vanishes, because every term will have ζ^2 as a factor.

17. In Art. 11 it will be found that the coefficient of z'^7 is

$$-\frac{6}{\lfloor 7}\{(n^2+n)^2-2(n^2+n)\}(2n+1),$$

and hence this term may be treated as the term was in the preceding Article.

18. Generally the coefficient of z'^r in U will be found to be

$$\frac{(n+2)(n+1)\ldots(n-r+4)}{\lfloor r}+(-1)^r\frac{(n-1)n\ldots(n+r-3)}{\lfloor r};$$

and hence, in order to carry on the process like that in Art. 16, we must show that this coefficient will take the form

$$\frac{2n+1}{\lfloor r}N,$$

where N is some rational integral function of n^2+n.

This may be etsablished inductively.

Assume that the required theorem holds for a certain value of r, and also for the value $r+1$; then it will hold for the value $r+2$.

For let it be assumed that

$$(n+2)(n+1)\ldots(n-r+4)+(-1)^r(n-1)n\ldots(n+r-3)=(2n+1)N_1,$$

and also that

$$(n+2)(n+1)\ldots(n-r+3)-(-1)^r(n-1)n\ldots(n+r-2)=(2n+1)N_2,$$

where N_1 and N_2 are rational integral functions of n^2+n; then we require to show that

$$(n+2)(n+1)\ldots(n-r+2)+(-1)^r(n-1)n\ldots(n+r-1)$$

will take a similar form.

We may denote our two assumed results thus:—

$$P+(-1)^rQ=(2n+1)N_1,$$
$$P(n-r+3)-(-1)^rQ(n+r-2)=(2n+1)N_2;$$

and then we have to investigate the form of

$$P(n-r+3)(n-r+2)+(-1)^r Q(n+r-2)(n+r-1).$$

Now the two following identities may be verified:—

$$(n-r+3)(n-r+2)=n^2+n-(r-3)(r-2)-2(n-r+3)(r-2),$$
$$(n+r-2)(n+r-1)=n^2+n-(r-3)(r-2)+2(n+r-2)(r-2).$$

Hence

$$\begin{aligned}P(n-r+3)&(n-r+2)+(-1)^r Q(n+r-2)(n+r-1)\\ &=\{n^2+n-(r-3)(r-2)\}(2n+1)N_1-2(r-2)(2n+1)N_2\\ &=(2n+1)N,\end{aligned}$$

where N is a rational integral function of n^2+n. Hence, as we have seen by actual inspection that for integral values of r up to 7 inclusive the required form is obtained, it follows that this form will be obtained for all positive integral values of r.

19. We may collect our results into two propositions, one of elementary algebra and one of the theory of Laplace's functions.

Let $f(z')$ stand for

$$\frac{1}{(n+3)r^{n+1}}\{(r+z')^{n+3}-r^{n+3}\}+\frac{r^n}{n-2}\{(r+z')^{-n+2}-r^{-n+2}\},$$

and suppose r greater than z', so as to ensure convergent series when the binomials are expanded in powers of z'; then the coefficient of every power of z' is the product of $2n+1$ into some rational integral function of n^2+n.

Let ζ' be a Laplace's function of the usual variables μ' and ψ', and ζ the same function of μ and ψ; and suppose r greater than ζ'; then

$$\Sigma\int_0^{4\pi} f(\zeta')\,P'_n\,d\omega'$$

is a function of ζ and its differential coefficients, which involves ζ^2 as a factor, and so vanishes when ζ vanishes.

20. I have not proposed to examine any difficulties which a reader may find in Poisson's process, but only to show that it can be made to furnish a general result instead of the result merely to the third order. Poisson's memoir has been much used by Bowditch in his translation of the 'Mécanique Céleste,' with a commentary (see vol. ii. p. 185); but Bowditch confines himself to the same order of approximation in the theorem as Poisson.

May 3, 1872.